ON THE GLOBAL WELL-POSEDNESS OF THE BOUSSINESQ SYSTEM WITH ZERO VISCOSITY

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Abstract. In this paper we prove the global well-posedness of the two-dimensional Boussinesq system with zero viscosity for rough initial data.

1. Introduction

This paper is a sequel to [12]. We continue to study the global existence for the two-dimensional Boussinesq system,

$$
(B_{\nu,\kappa})\begin{cases}\partial_t v + v\cdot\nabla v - \nu\Delta v + \nabla\pi = \theta e_2,\\ \partial_t\theta + v\cdot\nabla\theta - \kappa\Delta\theta = 0,\\ \operatorname{div} v = 0,\\ v_{|t=0} = v^0, \quad \theta_{|t=0} = \theta^0.\end{cases}
$$

Here, e_2 denotes the vector $(0,1)$, $v=(v_1,v_2)$ is the velocity field, π the scalar pressure and θ the temperature. The coefficients ν and κ are assumed to be positive; ν is called the kinematic viscosity and κ the molecular conductivity.

In the case of strictly positive coefficients ν and κ both velocity and temperature have sufficiently smoothing effects leading to the global well-posedness results proven by numerous authors in various function spaces (see [4, 9, 15] and the references therein).

For $\nu>0$ and $\kappa=0$ the problem of global well-posedness is well understood. In [5], Chae proved global well-posedness for initial data (v^0,θ^0) lying in Sobolev spaces $H^s\times H^s$, with $s>2$ (see also [14]). This result has been recently improved in [11] by taking the data in $H^s\times H^s$, with $s>0$. However we give only a global existence result without uniqueness in the energy space $L^2\times L^2$. In [1] we prove a uniqueness result for data belonging to $L^2\cap B^{-1}_{\infty,1}\times B^0_{2,1}$. More recently Danchin and Paicu [8] have established a uniqueness result in the energy space.

Our goal here is to study the global well-posedness of the system $(B_{0,\kappa})$, with $\kappa>0$. First of all, let us recall that the two-dimensional incompressible Euler system, corresponding to $\theta^0=0$, is globally well-posed in the Sobolev space H^s, with $s>2$. This is due to the advection of the vorticity by the flow: there is no accumulation of the vorticity and thus there is no finite time singularities according to B-K-M criterion [3]. In critical spaces like $B^{\frac{2}{p}+1}_{p,1}$ the situation is more complicate because we do not know if the B-K-M criterion works or not. In [16], Vishik proved that Euler system is globally well-posed in these critical Besov spaces. He used for the proof a new logarithmic estimate taking advantage on the particular structure of the vorticity equation in dimension two. For the Boussinesq system $(B_{0,\kappa})$, Chae has proved in [5] the global well-posedness for initial

data v^0, θ^0 lying in Soboloev space H^s, with $s > 2$, His method is basically related to Sobolev logarithmic estimate in which the velocity and the temperature are needed to be Lipschitzian explaining the restriction $s > 2$. We intend here to improve this result for rough initial data. Our results reads as follows (see the definition of Besov spaces given in next section).

Theorem 1.1. *Let $v^0 \in B_{p,1}^{1+\frac{2}{p}}$ be a divergence-free vector field of $\mathbb{R}^2$ and $\theta^0 \in L^r$, with $2 < r \leq p < \infty$. Then there exists a unique global solution (v, θ) to the Boussinesq system $(B_{0,\kappa}), \kappa > 0$ such that*

$$v \in \mathcal{C}(\mathbb{R}_+; B_{p,1}^{1+\frac{2}{p}}) \quad \textit{and} \quad \theta \in L^\infty(\mathbb{R}_+; L^r) \cap \widetilde{L}^1_{\text{loc}}(\mathbb{R}_+; B^2_{r,\infty}).$$

The situation in the case $p = +\infty$ is more subtle since Leray's projector is not continuous on L^∞ and we overcome this by working in homogeneous Besov spaces leading to more technical difficulties. Before stating our result we introduce the following sub-space of L^∞ :

$$u \in \mathcal{B}^\infty \Leftrightarrow \|u\|_{\mathcal{B}^\infty} := \|u\|_{L^\infty} + \|\Delta_{-1} u\|_{\dot{B}^0_{\infty,1}} < \infty.$$

We notice that $\mathcal{B}^\infty$ is a Banach space and independent of the choice of the unity dyadic partition. For the definition of the frequency localization operator Δ_{-1} we can see next section. Our second main result is the following:

Theorem 1.2. *Let $v^0 \in B^1_{\infty,1}$, with zero divergence and $\theta^0 \in \mathcal{B}^\infty$. Then there exists a unique global solution (v, θ) to the Boussinesq system $(B_{0,\kappa}), \kappa > 0$ such that*

$$v \in \mathcal{C}(\mathbb{R}_+; B^1_{\infty,1}) \quad \textit{and} \quad \theta \in L^\infty_{\text{loc}}(\mathbb{R}_+; \mathcal{B}^\infty) \cap \widetilde{L}^1_{\text{loc}}(\mathbb{R}_+; B^2_{\infty,\infty}).$$

The key of the proof is to bound for every time Lipschitz norm of both velocity and temperature. This will be done by using some logarithmic estimates and Theorem 3.1. The last one describes new smoothing effects for the transport-diffusion equation governed by a vector field which is not necessary Lipschitzian but only quasi-lipschitzian. Its proof is done in the spirit of [11].

The rest of this paper is organized as follows. In section 2, we recall some preliminary results on Besov spaces. Section 3 is devoted to the proof of smoothing effects. In section 4 and 5 we give respectively the proof of Theorem 1.1 and 1.2. We give in the appendix a logarithmic estimate and a commutator lemma.

2. Notation and preliminaries

Throughout this paper we shall denote by C some real positive constants which may be different in each occurrence and by C_0 a real positive constant depending on the initial data.

Let us introduce the so-called Littlewood-Paley decomposition and the corresponding cut-off operators. There exists two radial positive functions $\chi \in \mathcal{D}(\mathbb{R}^d)$ and $\varphi \in \mathcal{D}(\mathbb{R}^d \backslash \{0\})$ such that

i) $\chi(\xi) + \sum_{q \geq 0} \varphi(2^{-q}\xi) = 1; \quad \forall\, q \geq 1, \text{supp}\, \chi \cap \text{supp}\, \varphi(2^{-q}) = \varnothing$

ii) $\text{supp}\, \varphi(2^{-p}\cdot) \cap \text{supp}\, \varphi(2^{-q}\cdot) = \varnothing$, if $|p - q| \geq 2$.

For every $v \in \mathcal{S}'(\mathbb{R}^d)$ we set

$$\Delta_{-1}v = \chi(\mathrm{D})v \ ; \forall q \in \mathbb{N},\ \Delta_q v = \varphi(2^{-q}\mathrm{D})v \quad \text{and} \ \ S_q = \sum_{-1\leq p\leq q-1} \Delta_p.$$

The homogeneous operators are defined by

$$\dot{\Delta}_q v = \varphi(2^{-q}\mathrm{D})v, \quad \dot{S}_q v = \sum_{j\leq q-1} \dot{\Delta}_j v, \quad \forall q \in \mathbb{Z}.$$

From [2] we split the product uv into three parts:

$$uv = T_u v + T_v u + R(u,v),$$

with

$$T_u v = \sum_q S_{q-1}u\Delta_q v \quad \text{and} \quad R(u,v) = \sum_{|q'-q|\leq 1} \Delta_q u \Delta_{q'} v.$$

Let us now define inhomogeneous and homogeneous Besov spaces. For $(p,r) \in [1,+\infty]^2$ and $s \in \mathbb{R}$ we define the inhomogeneous Besov space $B^s_{p,r}$ as the set of tempered distributions u such that

$$\|u\|_{B^s_{p,r}} := \Big(2^{qs}\|\Delta_q u\|_{L^p}\Big)_{\ell^r} < +\infty.$$

The homogeneous Besov space $\dot{B}^s_{p,r}$ is defined as the set of $u \in \mathcal{S}'(\mathbb{R}^d)$ up to polynomials such that

$$\|u\|_{\dot{B}^s_{p,r}} := \Big(2^{qs}\|\dot{\Delta}_q u\|_{L^p}\Big)_{\ell^r(\mathbb{Z})} < +\infty.$$

Let $T > 0$ and $\rho \geq 1$, we denote by $L^\rho_T B^s_{p,r}$ the space of distributions u such that

$$\|u\|_{L^\rho_T B^s_{p,r}} := \Big\|\Big(2^{qs}\|\Delta_q u\|_{L^p}\Big)_{\ell^r}\Big\|_{L^\rho_T} < +\infty.$$

We say that u belongs to the space $\widetilde{L}^\rho_T B^s_{p,r}$ if

$$\|u\|_{\widetilde{L}^\rho_T B^s_{p,r}} := \Big(2^{qs}\|\Delta_q u\|_{L^\rho_T L^p}\Big)_{\ell^r} < +\infty.$$

The relations between these spaces are detailed below are a direct consequence of the Minkowski inequality. Let $\epsilon > 0$, then

$$L^\rho_T B^s_{p,r} \hookrightarrow \widetilde{L}^\rho_T B^s_{p,r} \hookrightarrow L^\rho_T B^{s-\epsilon}_{p,r}, \text{ if } \quad r \geq \rho,$$

$$L^\rho_T B^{s+\epsilon}_{p,r} \hookrightarrow \widetilde{L}^\rho_T B^s_{p,r} \hookrightarrow L^\rho_T B^s_{p,r}, \text{ if } \quad \rho \geq r.$$

We will make continuous use of Bernstein inequalities (see for example [6]).

Lemma 2.1. *There exists a constant C such that for $k \in \mathbb{N}$, $1 \leq a \leq b$ and for $u \in L^a(\mathbb{R}^d)$,*

$$\begin{aligned}
\sup_{|\alpha|=k} \|\partial^\alpha S_q u\|_{L^b} \quad &\leq \quad C^k\, 2^{q(k+d(\frac{1}{a}-\frac{1}{b}))}\|S_q u\|_{L^a},\\
C^{-k}2^{qk}\|\dot{\Delta}_q u\|_{L^a} \quad &\leq \quad \sup_{|\alpha|=k} \|\partial^\alpha \dot{\Delta}_q u\|_{L^a} \leq C^k 2^{qk}\|\dot{\Delta}_q u\|_{L^a}.
\end{aligned}$$

The following result is due to Vishik [16].

Lemma 2.2. *Let $d \geq 2$, there exists a positive constant C such that for any smooth function f and for any diffeomorphism ψ of $\mathbb{R}^d$ preserving Lebesgue measure, we have for all $p \in [1, +\infty]$ and for all $j, q \in \mathbb{Z}$,*

$$\|\dot{\Delta}_j(\dot{\Delta}_q f \circ \psi)\|_{L^p} \leq C 2^{-|j-q|} \|\nabla \psi^{\eta(j,q)}\|_{L^\infty} \|\dot{\Delta}_q f\|_{L^p},$$

with

$$\eta(j,q) = sign(j-q).$$

Let us now recall the following result proven in [8, 10].

Proposition 2.3. *Let $\nu \geq 0$, $(p,r) \in [1,\infty]^2$, $s \in]-1,1[$, $v \in L^1_{\text{loc}}(\mathbb{R}_+; \text{Lip}(\mathbb{R}^d))$ with zero divergence and f be a smooth fuction. Let a be any smooth solution of the transport-diffusion equation*

$$\partial_t a + v \cdot \nabla a - \nu \Delta a = f.$$

Then there is a constant $C : C(s,d)$ such that for every $t \in \mathbb{R}_+$

$$\|a\|_{\widetilde{L}^\infty_t B^s_{p,r}} + \nu^{\frac{1}{m}} \|a - \Delta_{-1} a\|_{\widetilde{L}^m_t B^{s+\frac{2}{m}}_{p,r}} \leq C e^{CV(t)} \Big(\|a^0\|_{B^s_{p,r}} + \int_0^t \|f(\tau)\|_{B^s_{p,r}} d\tau \Big),$$

where $V(t) := \displaystyle\int_0^t \|\nabla v(\tau)\|_{L^\infty} d\tau$.

3. Smoothing effects

This section is devoted to the proof of a new regularization effect for a transport-diffusion equation with respect to a vector field which is not necessary Lipschitzian. This problem was studied by the first author [11] in the context of singular vortex patches for two dimensional Navier-Stokes equations. The estimate given below is more precise.

Theorem 3.1. *Let v be a smooth divergence-free vector field of $\mathbb{R}^d$ with vorticity $\omega :=$ curl v. Let a be a smooth solution of the transport-diffusion equation*

$$\partial_t a + v \cdot \nabla a - \Delta a = 0; \quad a_{|t=0} = a^0.$$

Then we have for $q \in \mathbb{N} \cup \{-1\}$ and $t \geq 0$

$$2^{2q} \int_0^t \|\Delta_q a(\tau)\|_{L^\infty} d\tau \quad \lesssim \quad \|a^0\|_{L^\infty} \Big(1 + t + (q+2) \|\omega\|_{L^1_t L^\infty} + \|\nabla \Delta_{-1} v\|_{L^1_t L^\infty} \Big).$$

Remark 1. In [10], the first author proved in the case of Lipschitzian velocity the following estimate

$$2^{2q} \int_0^t \|\Delta_q a(\tau)\|_{L^\infty} d\tau \lesssim \|a^0\|_{L^\infty} \Big(1 + t + \int_0^t \|\nabla v(\tau)\|_{L^\infty} d\tau \Big). \tag{1}$$

We emphasize that the above theorem is also true when we change L^∞ by L^p, with $p \in [1, \infty]$.

Proof. The idea of the proof is the same as in [10]. We use Lagrangian formulation combined with intensive use of paradifferential calculus.
Let $q \in \mathbb{N}^*$, then the Fourier localized function $a_q := \Delta_q a$ satisfies

$$\partial_t a_q + S_{q-1} v \cdot \nabla a_q - \Delta a_q = (S_{q-1} - \text{Id}) v \cdot \nabla a_q - [\Delta_q, v \cdot \nabla] a := g_q. \tag{2}$$

Let ψ_q denote the flow of the regularized velocity $S_{q-1}v$:

$$\psi_q(t,x) = x + \int_0^t S_{q-1}v\big(\tau, \psi_q(\tau,x)\big)d\tau.$$

We set

$$\bar{a}_q(t,x) = a_q(t,\psi_q(t,x)) \quad \text{and} \quad \bar{g}_q(t,x) = g_q(t,\psi_q(t,x)).$$

From Leibnitz formula we deduce the following identity

$$\Delta\bar{a}_q(t,x) = \sum_{i=1}^{d} \Big\langle H_q \cdot (\partial^i\psi_q)(t,x), (\partial^i\psi_q)(t,x)\Big\rangle + (\nabla a_q)(t,\psi_q(t,x)) \cdot \Delta\psi_q(t,x),$$

where $H_q(t,x) := (\nabla^2 a_q)(t,\psi_q(t,x))$ is the Hessian matrix.
Straightforward computations based on the definition of the flow and Gronwall's inequality yield

$$\partial^i\psi_q(t,x) = e_i + h_q^i(t,x),$$

where $(e_i)_{i=1}^d$ is the canonical basis of $\mathbb{R}^d$ and the function h_q^i is estimated as follows

$$\|h_q^i(t)\|_{L^\infty} \lesssim V_q(t)e^{CV_q(t)}, \quad \text{with} \quad V_q(t) := \int_0^t \|\nabla S_{q-1}v(\tau)\|_{L^\infty} d\tau. \tag{3}$$

Applying Leibnitz formula and Bernstein inequality we find

$$\|\Delta\psi_q(t)\|_{L^\infty} \lesssim 2^q V_q(t)e^{CV_q(t)}. \tag{4}$$

The outcome is

$$\Delta\bar{a}_q(t,x) = (\Delta a_q)(t,\psi_q(t,x)) - \mathcal{R}_q(t,x), \tag{5}$$

with

$$\begin{aligned}
\|\mathcal{R}_q(t)\|_{L^\infty} &\lesssim \|\nabla a_q(t)\|_{L^\infty}\|\Delta\psi_q(t)\|_{L^\infty} \\
&+ \|\nabla^2 a_q(t)\|_{L^\infty} \sup_i \big(\|h_q^i(t)\|_{L^\infty} + \|h_q^i(t)\|_{L^\infty}^2\big) \\
&\lesssim 2^{2q}V_q(t)e^{CV_q(t)}\|a_q(t)\|_{L^\infty}.
\end{aligned} \tag{6}$$

In the last line we have used Bernstein inequality.
From (2) and (5) we see that $\bar{a}_q$ satisfies

$$(\partial_t - \Delta)\bar{a}_q(t,x) = \mathcal{R}_q(t,x) + \bar{g}_q(t,x).$$

Now, we will again localize in frequency this equation through the operator Δ_j. So we write from Duhamel formula,

$$\begin{aligned}
\Delta_j\bar{a}_q(t,x) &= e^{t\Delta}\Delta_j a_q(0) + \int_0^t e^{(t-\tau)\Delta}\Delta_j\mathcal{R}_q(\tau,x)d\tau \\
&+ \int_0^t e^{(t-\tau)\Delta}\Delta_j\bar{g}_q(\tau,x)d\tau.
\end{aligned} \tag{7}$$

At this stage we need the following lemma (see for instance [7]).

Lemma 3.2. *For $u \in L^\infty$ and $j \in \mathbb{N}$,*

$$\|e^{t\Delta}\Delta_j u\|_{L^\infty} \leq Ce^{-ct2^{2j}}\|\Delta_j u\|_{L^\infty}, \tag{8}$$

where the constants C and c depend only on the dimension d.

Combined with (6) this lemma yields, for every $j \in \mathbb{N}$,

$$\|e^{(t-\tau)\Delta}\Delta_j \mathcal{R}_q(\tau)\|_{L^\infty} \lesssim 2^{2q} V_q(\tau) e^{CV_q(\tau)} e^{-c(t-\tau)2^{2j}} \|a_q(\tau)\|_{L^\infty}. \tag{9}$$

Since the flow is an homeomorphism then we get again in view of Lemma 8

$$\begin{aligned}\|e^{(t-\tau)\Delta}\Delta_j \bar{g}_q(\tau)\|_{L^\infty} &\lesssim e^{-c(t-\tau)2^{2j}}\Big(\|[\Delta_q, v.\nabla]a(\tau)\|_{L^\infty} \\ &+ \|(S_{q-1}v - v)\cdot\nabla a_q\|_{L^\infty}\Big).\end{aligned} \tag{10}$$

From Proposition 5.4 we have

$$\begin{aligned}\|[\Delta_q, v\cdot\nabla]a(t)\|_{L^\infty} &\lesssim \|a(t)\|_{L^\infty}\Big(\|\nabla\Delta_{-1}v(t)\|_{L^\infty} + (q+2)\|\omega(t)\|_{L^\infty}\Big) \\ &\lesssim \|a^0\|_{L^\infty}\Big(\|\nabla\Delta_{-1}v(t)\|_{L^\infty} + (q+2)\|\omega(t)\|_{L^\infty}\Big).\end{aligned} \tag{11}$$

We have used in the last line the maximum principle: $\|a(t)\|_{L^\infty} \leq \|a^0\|_{L^\infty}$.
On the other hand since $q \in \mathbb{N}^*$, we can easily obtain

$$\begin{aligned}\|(S_{q-1}v - v)\cdot\nabla a_q\|_{L^\infty} &\lesssim \|a_q\|_{L^\infty} 2^q \sum_{j\geq q-1} 2^{-j}\|\Delta_j\omega\|_{L^\infty} \\ &\lesssim \|a^0\|_{L^\infty}\|\omega\|_{L^\infty}.\end{aligned} \tag{12}$$

Putting together (7), (9), (10), (11) and (12) we find

$$\begin{aligned}\|\Delta_j\bar{a}_q(t)\|_{L^\infty} &\lesssim e^{-ct2^{2j}}\|\Delta_j a_q^0\|_{L^\infty} \\ &+ V_q(t)e^{CV_q(t)}2^{2q}\int_0^t e^{-c(t-\tau)2^{2j}}\|a_q(\tau)\|_{L^\infty}d\tau \\ &+ (q+2)\|a^0\|_{L^\infty}\int_0^t e^{-c(t-\tau)2^{2j}}\|\omega(\tau)\|_{L^\infty}d\tau \\ &+ \|a^0\|_{L^\infty}\int_0^t e^{-c(t-\tau)2^{2j}}\|\nabla\Delta_{-1}v(\tau)\|_{L^\infty}d\tau.\end{aligned}$$

Integrating in time and using Young inequalities, we obtain for all $j \in \mathbb{N}$

$$\begin{aligned}\|\Delta_j\bar{a}_q\|_{L^1_tL^\infty} &\lesssim (2^{2j})^{-1}\Big(\|\Delta_j a_q^0\|_{L^\infty} + (q+2)\|a^0\|_{L^\infty}\|\omega\|_{L^1_tL^\infty} + \\ &\|a^0\|_{L^\infty}\|\nabla\Delta_{-1}v\|_{L^1_tL^\infty}\Big) + V_q(t)e^{CV_q(t)}2^{2(q-j)}\|a_q\|_{L^1_tL^\infty}.\end{aligned}$$

Let N be a large integer that will be chosen later. Since the flow is an homeomorphism, then we can write

$$\begin{aligned}2^{2q}\|a_q\|_{L^1_tL^\infty} &= 2^{2q}\|\bar{a}_q\|_{L^1_tL^\infty} \\ &\leq 2^{2q}\Big(\sum_{|j-q|<N}\|\Delta_j\bar{a}_q\|_{L^1_tL^\infty} + \sum_{|j-q|\geq N}\|\Delta_j\bar{a}_q\|_{L^1_tL^\infty}\Big).\end{aligned}$$

Hence, for all $q > N$, one has

$$\begin{aligned} 2^{2q}\|a_q\|_{L^1_tL^\infty} &\lesssim \|a^0\|_{L^\infty} + 2^{2N}\|a^0\|_{L^\infty}\Big((q+2)\|\omega\|_{L^1_tL^\infty} + \|\nabla\Delta_{-1}v\|_{L^1_tL^\infty}\Big) \\ &+ V_q(t)e^{CV_q(t)}2^{2N}2^{2q}\|a_q\|_{L^1_tL^\infty} + 2^{2q}\sum_{|j-q|\geq N}\|\Delta_j\bar{a}_q\|_{L^1_tL^\infty}. \end{aligned}$$

According to Lemma 2.2, we have

$$\|\Delta_j\bar{a}_q(t)\|_{L^\infty} \lesssim 2^{-|q-j|}e^{CV_q(t)}\|a_q(t)\|_{L^\infty}.$$

Thus, we infer

$$\begin{aligned} 2^{2q}\|a_q\|_{L^1_tL^\infty} &\lesssim \|a^0\|_{L^\infty} + 2^{2N}\|a^0\|_{L^\infty}\Big((q+2)\|\omega\|_{L^1_tL^\infty} + \|\nabla\Delta_{-1}v\|_{L^1_tL^\infty}\Big) \\ &+ V_q(t)e^{CV_q(t)}2^{2N}2^{2q}\|a_q\|_{L^1_tL^\infty} + 2^{-N}e^{CV_q(t)}2^{2q}\|a_q\|_{L^1_tL^\infty}. \end{aligned}$$

For low frequencies, $q \leq N$, we write

$$2^{2q}\|a_q\|_{L^1_tL^\infty} \lesssim 2^{2N}\|a\|_{L^1_tL^\infty}.$$

Therefore we get for $q \in \mathbb{N}\cup\{-1\}$,

$$\begin{aligned} 2^{2q}\|a_q\|_{L^1_tL^\infty} &\lesssim \|a^0\|_{L^\infty} + 2^{2N}\|a\|_{L^1_tL^\infty} \\ &+ 2^{2N}\|a^0\|_{L^\infty}\Big((q+2)\|\omega\|_{L^1_tL^\infty} + \|\nabla\Delta_{-1}v\|_{L^1_tL^\infty}\Big) \\ &+ \Big(V_q(t)e^{CV_q(t)}2^{2N} + 2^{-N}e^{CV_q(t)}\Big)2^{2q}\|a_q\|_{L^1_tL^\infty}. \end{aligned}$$

Choosing N and t such that

$$V_q(t)e^{CV_q(t)}2^{2N} + e^{CV_q(t)}2^{-N} \lesssim \epsilon,$$

where $\epsilon << 1$. This is possible for small time t such that

$$V_q(t) \leq C_1,$$

where C_1 is a small absolute constant.
Under this assumption, one obtains for $q \geq -1$

$$2^{2q}\|a_q\|_{L^1_tL^\infty} \lesssim \|a\|_{L^1_tL^\infty} + \|a^0\|_{L^\infty}\Big(1 + (q+2)\|\omega\|_{L^1_tL^\infty} + \|\nabla\Delta_{-1}v\|_{L^1_tL^\infty}\Big).$$

Let us now see how to extend this for arbitrarly large time T. We take a partition $(T_i)_{i=1}^M$ of $[0,T]$ such that

$$\int_{T_i}^{T_{i+1}}\|\nabla S_{q-1}v(t)\|_{L^\infty}dt \simeq C_1.$$

Reproducing the same arguments as above we find in view of $\|a(T_i)\|_{L^\infty} \leq \|a^0\|_{L^\infty}$,

$$\begin{aligned}2^{2q}\int_{T_i}^{T_{i+1}}\|a_q(t)\|_{L^\infty}dt &\lesssim \int_{T_i}^{T_{i+1}}\|a(t)\|_{L^\infty}dt+\|a^0\|_{L^\infty}\\ &+ \|a^0\|_{L^\infty}\Big((q+2)\int_{T_i}^{T_{i+1}}\|\omega(t)\|_{L^\infty}dt\\ &+ \int_{T_i}^{T_{i+1}}\|\nabla\Delta_{-1}v(t)\|_{L^\infty}dt\Big).\end{aligned}$$

Summing these estimates we get

$$\begin{aligned}2^{2q}\|a_q\|_{L^1_TL^\infty} &\lesssim \|a\|_{L^1_TL^\infty}+(M+1)\|a^0\|_{L^\infty}+\\ &+ \|a^0\|_{L^\infty}\Big((q+2)\|\omega\|_{L^1_TL^\infty}+\|\nabla\Delta_{-1}v\|_{L^1_TL^\infty}\Big).\end{aligned}$$

As $M\approx V_q(T)$, then

$$\begin{aligned}2^{2q}\|a_q\|_{L^1_TL^\infty} &\lesssim \|a\|_{L^1_TL^\infty}+(V_q(T)+1)\|a^0\|_{L^\infty}+\\ &+ \|a^0\|_{L^\infty}\Big((q+2)\|\omega\|_{L^1_TL^\infty}+\|\nabla\Delta_{-1}v\|_{L^1_TL^\infty}\Big).\end{aligned}$$

Since

$$\|\nabla S_{q-1}v\|_{L^\infty}\leq\|\nabla\Delta_{-1}v\|_{L^\infty}+(q+2)\|\omega\|_{L^\infty},$$

then inserting this estimate into the previous one

$$2^{2q}\|a_q\|_{L^1_TL^\infty} \lesssim \|a^0\|_{L^\infty}\Big((1+T)+(q+2)\|\omega\|_{L^1_TL^\infty}+\|\nabla\Delta_{-1}v\|_{L^1_TL^\infty}\Big).$$

This is the desired result. □

4. Proof of Theorem 1.1

We restrict ourselves to the *a priori* estimates. The existence and uniqueness parts are easily obtained with small modifications of the proof of Theorem 1.2.

Proposition 4.1. *For $v^0\in B_{p,1}^{1+\frac{2}{p}}$ and $\theta^0\in L^r$, with $2<r\leq p$, we have for $t\in\mathbb{R}_+$*

1)
$$\|\theta(t)\|_{L^r}\leq\|\theta^0\|_{L^r}.$$

2)
$$\|\theta\|_{L^1_tB_{r,1}^{1+\frac{2}{r}}}+\|\omega(t)\|_{L^\infty}+\|\omega\|_{\widetilde{L}^\infty_tB^0_{\infty,1}}\leq C_0e^{e^{C_0t}}.$$

3)
$$\|\theta\|_{\widetilde{L}^1_tB^2_{r,\infty}}+\|v\|_{\widetilde{L}^\infty_tB_{p,1}^{1+\frac{2}{p}}}\leq C_0e^{e^{e^{C_0t}}},$$

where the constant C_0 depends on the quantity $\|\theta^0\|_{L^r}$ and $\|v^0\|_{B_{p,1}^{1+\frac{2}{p}}}$.

Proof. The first estimate can be easily obtained from L^r energy estimate. for the second one, we recall that the vorticity $\omega = \partial_1 v^2 - \partial_2 v^1$ satisifies the equation

$$\partial_t \omega + v \cdot \nabla \omega = \partial_1 \theta. \tag{13}$$

Taking the L^∞ norm we get

$$\|\omega(t)\|_{L^\infty} \leq \|\omega^0\|_{L^\infty} + \|\nabla\theta\|_{L^1_t L^\infty}. \tag{14}$$

Using the embedding $B^{1+\frac{2}{r}}_{r,1} \hookrightarrow \mathrm{Lip}(\mathbb{R}^2)$ we obtain

$$\|\omega(t)\|_{L^\infty} \lesssim \|\omega^0\|_{L^\infty} + \|\theta\|_{L^1_t B^{1+\frac{2}{r}}_{r,1}}.$$

From Theorem 3.1 applied to the temperature equation and by Bernstein inequalities we deduce for $\epsilon > 0$,

$$\begin{aligned}
\|\theta\|_{L^1_t B^{2-\epsilon}_{r,\infty}} &\lesssim \|\theta^0\|_{L^r}\Big(1 + t + \|\omega\|_{L^1_t L^\infty} + \|\Delta_{-1}\nabla v\|_{L^1_t L^\infty}\Big)\\
&\lesssim \|\theta^0\|_{L^r}\Big(1 + t + \|\omega\|_{L^1_t L^\infty} + \|\nabla v\|_{L^1_t L^p}\Big).
\end{aligned}$$

This leads for $r > 2$ to the inequality

$$\|\theta\|_{L^1_t B^{1+\frac{2}{r}}_{r,\infty}} \lesssim \|\theta^0\|_{L^r}\Big(1 + t + \|\omega\|_{L^1_t L^\infty} + \|\nabla v\|_{L^1_t L^p}\Big).$$

On the other hand we have the classical result $\|\nabla v\|_{L^p} \approx \|\omega\|_{L^p}$, for $p \in]1, \infty[$. Thus we get

$$\|\theta\|_{L^1_t B^{1+\frac{2}{r}}_{r,1}} \lesssim \|\theta^0\|_{L^r}\Big(1 + t + \|\omega\|_{L^1_t L^\infty} + \|\omega\|_{L^1_t L^p}\Big).$$

The estimate of the L^p norm of the vorticity can be done as its L^∞ norm $(r \leq p)$

$$\|\omega(t)\|_{L^p} \lesssim \|\omega^0\|_{L^p} + \|\theta\|_{L^1_t B^{1+\frac{2}{r}}_{r,1}}.$$

Set $f(t) := \|\omega(t)\|_{L^\infty \cap L^p} + \|\theta\|_{L^1_t B^{1+\frac{2}{r}}_{r,1}}$. Then combining the above estimates yields

$$f(t) \lesssim \|\omega^0\|_{L^\infty \cap L^p} + \|\theta^0\|_{L^r}(1+t) + \|\theta^0\|_{L^r}\int_0^t f(\tau)d\tau.$$

According to Gronwall's inequality, one has

$$\|\omega(t)\|_{L^\infty \cap L^p} + \|\theta\|_{L^1_t B^{1+\frac{2}{r}}_{r,1}} \lesssim \Big(\|\omega^0\|_{L^\infty \cap L^p} + \|\theta^0\|_{L^r}(1+t)\Big)e^{Ct\|\theta^0\|_{L^r}} \leq C_0 e^{C_0 t}, \tag{15}$$

where C_0 is a constant depending on the initial data.

Let us now turn to the estimate of $\|\omega(t)\|_{B^0_{\infty,1}}$. From Proposition 5.3 and Besov embeddings,

$$\begin{aligned}
\|\omega\|_{\widetilde{L}^\infty_t B^0_{\infty,1}} &\lesssim \big(\|\omega^0\|_{L^\infty} + \|\theta\|_{L^1_t B^1_{\infty,1}}\big)\big(1 + \|\nabla v\|_{L^1_t L^\infty}\big)\\
&\lesssim \big(\|\omega^0\|_{L^\infty} + \|\theta\|_{L^1_t B^{1+\frac{2}{r}}_{r,1}}\big)\big(1 + \|\nabla v\|_{L^1_t L^\infty}\big).
\end{aligned} \tag{16}$$

On the other hand we have

$$\begin{aligned}\|\nabla v(t)\|_{L^\infty} &\leq \|\nabla\Delta_{-1}v(t)\|_{L^\infty}+\sum_{q\in\mathbb{N}}\|\Delta_q\nabla v(t)\|_{L^\infty}\\ &\lesssim \|\nabla\Delta_{-1}v(t)\|_{L^p}+\|\omega(t)\|_{B^0_{\infty,1}}\\ &\lesssim \|\omega(t)\|_{L^p}+\|\omega\|_{\widetilde{L}^\infty_t B^0_{\infty,1}}. \qquad (17)\end{aligned}$$

Putting together (15), (16) and (17) and using Gronwall's inequality gives

$$\|\nabla v\|_{L^\infty}+\|\omega\|_{\widetilde{L}^\infty_t B^0_{\infty,1}}\leq C_0 e^{e^{C_0 t}}. \qquad (18)$$

It remains to prove the third point of the proposition. The smoothing effect on θ is a direct consequence of (1) and the above inequality,

$$\|\theta\|_{\widetilde{L}^1_t B^2_{r,\infty}}\leq C_0 e^{e^{C_0 t}}.$$

Concerning the velocity estimate we write

$$\|v\|_{\widetilde{L}^\infty_t B^{1+\frac{2}{p}}_{p,1}}\lesssim \|v\|_{L^\infty_t L^p}+\|\omega\|_{\widetilde{L}^\infty_t B^{\frac{2}{p}}_{p,1}}.$$

Using the velocity equation, we obatin

$$\|v(t)\|_{L^p}\leq \|v^0\|_{L^p}+t\|\theta^0\|_{L^p}+\int_0^t\|\mathcal{P}(v\cdot\nabla v)(\tau)\|_{L^p}d\tau.$$

where $\mathcal{P}$ denotes Leray projector. It follows from classical estimate that

$$\|\mathcal{P}(v\cdot\nabla v)\|_{L^p}\lesssim\|v\cdot\nabla v\|_{L^p}\lesssim\|v\|_{L^p}\|\nabla v\|_{L^\infty}.$$

Thus we get in view of Gronwall's inequality and (18)

$$\|v\|_{L^\infty_t L^p}\leq C_0 e^{e^{e^{C_0 t}}}. \qquad (19)$$

It remains to estimate $\|\omega(t)\|_{B^{\frac{2}{p}}_{p,1}}$. We apply Proposition 2.3 to the vorticity equation and we use Besov embeddings,

$$\begin{aligned}\|\omega\|_{\widetilde{L}^\infty_t B^{\frac{2}{p}}_{p,1}} &\lesssim e^{CV(t)}(\|\omega^0\|_{B^{\frac{2}{p}}_{p,1}}+\|\theta\|_{L^1_t B^{1+\frac{2}{p}}_{p,1}})\\ &\lesssim e^{CV(t)}(\|\omega^0\|_{B^{\frac{2}{p}}_{p,1}}+\|\theta\|_{L^1_t B^{1+\frac{2}{r}}_{r,1}}).\end{aligned}$$

It suffices now to use (15) and (18). $\square$

5. Proof of Theorem 1.2

The case $p=+\infty$ is more subtle and the difficulty comes from the term $\|\nabla\Delta_{-1}v\|_{L^\infty}$, since Riesz transforms do not map L^∞ to itself. To avoid this problem we use a frequency interpolation method. The proof will be done in several steps. The first one deals with some a priori estimates. We give in the second the uniqueness result and the last is reserved to the proof of the existence part.

5.1. **A priori estimates.** The main result of this section is the following:

Proposition 5.1. *There exists a constant C_0 depending on $\|v^0\|_{B^0_{\infty,1}}$ and $\|\theta^0\|_{L^\infty}$ such that for $t\in[0,\infty[$*

$$\begin{aligned}\|\theta(t)\|_{L^\infty}\leq\|\theta^0\|_{L^\infty};\quad \|\theta\|_{L^1_tB^1_{\infty,1}}&\leq C_0e^{C_0t^3}\quad\textit{and}\\ \|v\|_{\widetilde{L}^\infty_tB^1_{\infty,1}}+\|\theta\|_{\widetilde{L}^1_tB^2_{\infty,\infty}}+\|\theta(t)\|_{\mathcal{B}^\infty}&\leq C_0e^{e^{C_0t^3}}.\end{aligned}$$

Proof. The L^∞-bound of the temperature can be easily obtained from the maximum principle. To give the other bounds we start with the following estimate for the vorticity, which is again a direct consequence of the maximum principle,

$$\|\omega(t)\|_{L^\infty}\leq\|\omega^0\|_{L^\infty}+\|\nabla\theta\|_{L^1_tL^\infty}\leq\|\omega^0\|_{L^\infty}+\|\theta\|_{L^1_tB^1_{\infty,1}}. \tag{20}$$

Let $N\in\mathbb{N}^*$, then we get by definition of Besov spaces and the maximum principle

$$\begin{aligned}\|\theta\|_{L^1_tB^1_{\infty,1}}&=\sum_{q\leq N-1}2^q\|\Delta_q\theta\|_{L^1_tL^\infty}+\sum_{q\geq N}2^q\|\Delta_q\theta\|_{L^1_tL^\infty}\\&\lesssim 2^Nt\|\theta^0\|_{L^\infty}+\sum_{q\geq N}2^q\|\Delta_q\theta\|_{L^1_tL^\infty}.\end{aligned}$$

By virtue of Theorem 3.1 one has

$$\begin{aligned}\|\theta\|_{L^1_tB^1_{\infty,1}}&\lesssim 2^Nt\|\theta^0\|_{L^\infty}+2^{-N}\|\theta^0\|_{L^\infty}\Big(1+t+\|\nabla\Delta_{-1}v\|_{L^1_tL^\infty}+N\|\omega\|_{L^1_tL^\infty}\Big)\\&\lesssim 2^Nt\|\theta^0\|_{L^\infty}+2^{-N}\|\theta^0\|_{L^\infty}\Big(1+t+\|\nabla\Delta_{-1}v\|_{L^1_tL^\infty}\Big)+\|\omega\|_{L^1_tL^\infty}.\end{aligned}$$

Choosing judiciously N we get

$$\|\theta\|_{L^1_tB^1_{\infty,1}}\lesssim\|\omega\|_{L^1_tL^\infty}+t^{\frac12}\|\theta^0\|_{L^\infty}\Big(1+t+\|\nabla\Delta_{-1}v\|_{L^1_tL^\infty}\Big)^{\frac12} \tag{21}$$

The following lemma gives an estimate of the low frequency of the velocity.

Lemma 5.2. *For all $t\geq0$, wed have*

$$\|\nabla\Delta_{-1}v(t)\|_{L^\infty}\lesssim 1+\log\big(e+\|v^0\|_{L^\infty}+t\|\theta^0\|_{L^\infty}\big)\|\omega\|_{L^\infty_tL^\infty}+t\|\omega\|^2_{L^\infty_tL^\infty}.$$

Proof. Fix $N\in\mathbb{N}^*$. Since $\Delta_{-1}=\Delta_{-1}(\dot S_{-N}+\sum_{q=-N}^0\dot\Delta_q)$ then we have

$$\begin{aligned}\|\nabla\Delta_{-1}v\|_{L^\infty}&\lesssim\|\nabla\dot S_{-N}v\|_{L^\infty}+\sum_{q=-N}^0\|\nabla\dot\Delta_qv\|_{L^\infty}\\&\lesssim 2^{-N}\|v\|_{L^\infty}+\sum_{-N}^0\|\dot\Delta_q\omega\|_{L^\infty}\\&\lesssim 2^{-N}\|v\|_{L^\infty}+N\|\omega\|_{L^\infty}.\end{aligned}$$

Taking $N\approx\log(e+\|v\|_{L^\infty})$ we get

$$\|\nabla\Delta_{-1}v\|_{L^\infty}\lesssim 1+\|\omega\|_{L^\infty}\log(e+\|v\|_{L^\infty}). \tag{22}$$

It remains to estimate $\|v\|_{L^\infty}$. Let $M\in\mathbb{N}$ then we have

$$\|v\|_{L^\infty}\lesssim\|\dot{S}_{-M}v\|_{L^\infty}+2^M\|\omega\|_{L^\infty}.$$

Now using the equation of the velocity we get

$$\begin{aligned}\|\dot{S}_{-M}v(t)\|_{L^\infty} &\leq \|\dot{S}_{-M}v^0\|_{L^\infty}+\|\dot{S}_{-M}\theta\|_{L^1_tL^\infty}\\ &+ \int_0^t\|\dot{S}_{-M}\mathrm{div}\,\mathcal{P}(v\otimes v)(\tau)\|_{L^\infty}d\tau\\ &\lesssim \|v^0\|_{L^\infty}+t\|\theta^0\|_{L^\infty}+2^{-M}\int_0^t\|v(\tau)\|^2_{L^\infty}d\tau.\end{aligned}$$

We have used the following inequality

$$\|\dot{S}_{-M}\mathrm{div}\,\mathcal{P}(v\otimes v)\|_{L^\infty}\leq\sum_{q\leq -M-1}\|\dot{\Delta}_q\mathrm{div}\,\mathcal{P}(v\otimes v)\|_{L^\infty}\lesssim\sum_{q\leq -M-1}2^q\|v\otimes v\|_{L^\infty}.$$

Thus we obtain

$$\|v\|_{L^\infty}\lesssim\|v^0\|_{L^\infty}+t\|\theta^0\|_{L^\infty}+2^{-M}\int_0^t\|v(\tau)\|^2_{L^\infty}d\tau+2^M\|\omega(t)\|_{L^\infty}.$$

Taking M such that

$$2^{2M}\approx\frac{\int_0^t\|v\|^2_{L^\infty}d\tau}{\|\omega\|_{L^\infty}},$$

we find

$$\|v\|_{L^\infty}\lesssim\|v^0\|_{L^\infty}+t\|\theta^0\|_{L^\infty}+\|\omega(t)\|^{\frac12}_{L^\infty}\Big(\int_0^t\|v(\tau)\|^2_{L^\infty}d\tau\Big)^{\frac12}.$$

According to Gronwall's inequality we get

$$\|v\|_{L^\infty}\lesssim\big(\|v^0\|_{L^\infty}+t\|\theta^0\|_{L^\infty}\big)e^{Ct\|\omega\|_{L^\infty_tL^\infty}}. \tag{23}$$

Inserting this estimate into (22) we find the desired inequality. □

Lemma 5.2 and (21) yield

$$\begin{aligned}\|\theta\|^2_{L^1_tB^1_{\infty,1}} &\leq C_0(1+t^2)+\|\omega\|^2_{L^1_tL^\infty}+C_0(1+t^2)\int_0^t\|\omega\|^2_{L^\infty_\tau L^\infty}d\tau\\ &\leq C_0(1+t^2)\Big(1+\int_0^t\|\omega\|^2_{L^\infty_\tau L^\infty}d\tau\Big).\end{aligned}$$

Combining this estimate with (20) yields

$$\|\omega\|^2_{L^\infty_tL^\infty}\leq C_0(1+t^2)\Big(1+\int_0^t\|\omega\|^2_{L^\infty_\tau L^\infty}d\tau\Big).$$

Applying Gronwall's inequality we get

$$\|\omega(t)\|_{L^\infty}\leq C_0e^{C_0t^3}. \tag{24}$$

This gives

$$\|\theta\|_{L^1_tB^1_{\infty,1}}\leq C_0e^{C_0t^3}. \tag{25}$$

From Lemma 5.2 we have

$$\|\nabla\Delta_{-1}v(t)\|_{L^\infty}\leq C_0e^{C_0t^3}. \tag{26}$$

Let us now turn to the estimate of the vorticity in $B^0_{\infty,1}$ space. For this purpose we apply Proposition 5.3 to the vorticity equation, with $p=+\infty$ and $r=1$

$$\|\omega\|_{\widetilde{L}^\infty_t B^0_{\infty,1}}\lesssim\big(\|\omega^0\|_{L^\infty}+\|\nabla\theta\|_{L^1_tB^0_{\infty,1}}\big)\Big(1+\int_0^t\|\nabla v(\tau)\|_{L^\infty}d\tau\Big). \tag{27}$$

On the other hand we have by definition and from (26) and (27)

$$\begin{aligned}\|\nabla v(t)\|_{L^\infty}\lesssim\|v\|_{\widetilde{L}^\infty_tB^1_{\infty,1}}&\lesssim\|\nabla\Delta_{-1}v\|_{L^\infty_tL^\infty}+\sum_{q\in\mathbb{N}}\|\Delta_q\omega\|_{L^\infty_tL^\infty}\\&\lesssim C_0e^{C_0t^3}+\|\omega\|_{\widetilde{L}^\infty_tB^0_{\infty,1}}\\&\lesssim C_0e^{C_0t^3}\Big(1+\int_0^t\|v(\tau)\|_{B^1_{\infty,1}}d\tau\Big).\end{aligned}$$

It suffices now to use Gronwall's inequality.

To estimate $\|\theta\|_{L^1_tB^2_{\infty,\infty}}$ it suffices to combine (1) with the Lipschitz estimate of the velocity. The last estimate $\|\theta(t)\|_{\mathcal{B}^\infty}$ will be done as follows:

$$\|\theta(t)\|_{\mathcal{B}^\infty}\leq\|\theta^0\|_{L^\infty}+\sum_{q\leq 0}\|\dot{\Delta}_q\theta(t)\|_{L^\infty}.$$

Using the temperature equation we find

$$\begin{aligned}\|\dot{\Delta}_q\theta(t)\|_{L^\infty}&\leq\|\dot{\Delta}_q\theta^0\|_{L^\infty}+\|\dot{\Delta}_q(v\cdot\nabla\theta)\|_{L^1_tL^\infty}+\|\dot{\Delta}_q\Delta\theta\|_{L^1_tL^\infty}\\&\lesssim\|\dot{\Delta}_q\theta^0\|_{L^\infty}+2^q\|v\,\theta\|_{L^1_tL^\infty}+2^{2q}\|\theta\|_{L^1_tL^\infty}\\&\lesssim\|\dot{\Delta}_q\theta^0\|_{L^\infty}+2^q\|\theta^0\|_{L^\infty}\|v\|_{L^1_tL^\infty}+2^{2q}t\|\theta^0\|_{L^\infty}.\end{aligned}$$

Therefore we get

$$\sum_{q\leq 0}\|\dot{\Delta}_q\theta(t)\|_{L^\infty}\lesssim\sum_{q\leq 0}\|\dot{\Delta}_q\theta^0\|_{L^\infty}+C_0e^{e^{C_0t^3}}.$$

This concludes the proof of the proposition. □

5.2. **Uniqueness part.** As it was shown in the previous paragraph we can give an *a priori* estimates for both Lipschitz norms of the velocity and the temperature only under the assumption $v^0\in B^0_{\infty,1}$ and $\theta^0\in L^\infty$. However it seems that the uniqueness part (even the existence) needs an addition condition of the initial data θ^0, namely $\Delta_{-1}\in\dot{B}^0_{\infty,1}$. Let us consider two solutions $\{(v^j,\theta^j)\}_{j=1}^2$ for the system $(B_{0,\kappa})$, with initial data $(v^{j,0},\theta^{j,0})$, $j=1,2$ and satisfying for a fixed time $T>0$

$$v^j\in L^\infty_T\,B^1_{\infty,1}\qquad\text{and}\qquad\theta^j\in L^\infty_T\mathcal{B}^\infty\cap L^1_T\,\mathrm{Lip}(\mathbb{R}^2).$$

We set

$$v=v^1-v^2,\ \theta=\theta^1-\theta^2,\ \pi=\pi^1-\pi^2,\ v^0=v^{1,0}-v^{2,0},\quad\text{and}\quad\theta^0=\theta^{1,0}-\theta^{2,0}.$$

Thus we have the equations

$$\partial_t v + v^1\cdot\nabla v = -\nabla\pi - v\cdot\nabla v^2 + \theta e_2, \tag{28}$$

$$\partial_t\theta + v^1\cdot\nabla\theta - \Delta\theta = -v\cdot\nabla\theta^2. \tag{29}$$

Hereafter we denote $V_j(t) := C\|\nabla v^j\|_{L^1_tL^\infty}, j=1,2$. Now applying Lemma 2.3 yields

$$\|v(t)\|_{B^0_{\infty,1}} \lesssim e^{V_1(t)}\Big(\|v^0\|_{B^0_{\infty,1}} + \int_0^t \big(\|\nabla\pi(\tau)\|_{B^0_{\infty,1}} + \|v\cdot\nabla v^2(\tau)\|_{B^0_{\infty,1}} + \|\theta(\tau)\|_{B^0_{\infty,1}}\big)d\tau\Big).$$

To estimate the pressure we write the following identity

$$\Delta\pi = -\mathrm{div}\,\big(v^1\cdot\nabla v + v\cdot\nabla v^2\big) + \partial_2\theta.$$

Since $\mathrm{div}\,\big(v^1\cdot\nabla v\big) = \mathrm{div}\,\big(v\cdot\nabla v^1\big)$ then

$$\nabla\pi = -\nabla\Delta^{-1}\mathrm{div}\,\big(v\cdot\nabla(v^1+v^2)\big) + \nabla\Delta^{-1}\partial_2\theta.$$

From the embedding $\dot B^0_{\infty,1}\hookrightarrow B^0_{\infty,1}$ and the fact that Riesz transforms act continuously on homogeneous Besov spaces, one obtains

$$\|\nabla\pi\|_{B^0_{\infty,1}} \lesssim \|v\cdot\nabla(v^1+v^2)\|_{\dot B^0_{\infty,1}} + \|\theta\|_{\dot B^0_{\infty,1}}.$$

It is easy to see that

$$\begin{aligned}\|v\cdot\nabla(v^1+v^2)\|_{\dot B^0_{\infty,1}} &\lesssim \sum_{q\le 0} 2^q\|\dot\Delta_q(v\otimes(v^1+v^2))\|_{L^\infty} + \sum_{q>0}\|\Delta_q(v\cdot\nabla(v^1+v^2))\|_{L^\infty}\\ &\lesssim \|v\|_{L^\infty}\|v^1+v^2\|_{L^\infty} + \|v\cdot\nabla(v^1+v^2)\|_{B^0_{\infty,1}}.\end{aligned}$$

Using Bony's decomposition and the incompressibilty of the velocity v one obtains the general estimate

$$\|v\cdot\nabla w\|_{B^0_{\infty,1}} \lesssim \|v\|_{B^0_{\infty,1}}\|w\|_{B^1_{\infty,1}}.$$

It follows that

$$\begin{aligned}\|v\cdot\nabla(v^1+v^2)\|_{B^0_{\infty,1}} &\lesssim \|v\|_{B^0_{\infty,1}}\|v^1+v^2\|_{B^1_{\infty,1}}\\ \|v\cdot\nabla v^2\|_{B^0_{\infty,1}} &\lesssim \|v\|_{B^0_{\infty,1}}\|v^2\|_{B^1_{\infty,1}}.\end{aligned}$$

Putting together these estimates gives

$$\|v(t)\|_{B^0_{\infty,1}} \lesssim e^{V_1(t)}\Big(\|v^0\|_{B^0_{\infty,1}} + \|\theta\|_{L^1_t\dot B^0_{\infty,1}} + \int_0^t \|v(\tau)\|_{B^0_{\infty,1}} w_{1,2}(\tau)d\tau\Big), \tag{30}$$

with

$$w_{1,2}(t) := \|v^1(t)\|_{B^1_{\infty,1}} + \|v^2(t)\|_{B^1_{\infty,1}}.$$

It remains to estimate the quantity $\|\theta\|_{L^1_t\dot B^0_{\infty,1}}$. For this aim we split θ into low and high frequencies

$$\|\theta\|_{\dot B^0_{\infty,1}} \le \sum_{q\le 0}\|\dot\Delta_q\theta\|_{L^\infty} + \|\theta - \Delta_{-1}\theta\|_{B^0_{\infty,1}}.$$

Using the equation of θ we get easily

$$\begin{aligned}\|\dot\Delta_q\theta(t)\|_{L^\infty} &\leq \|\dot\Delta_q\theta^0\|_{L^\infty}+\|\dot\Delta_q(\mathrm{div}\,(v^1\,\theta+v\,\theta^2)\|_{L^1_tL^\infty}+\|\dot\Delta_q\Delta\theta\|_{L^1_tL^\infty}\\ &\lesssim \|\dot\Delta_q\theta^0\|_{L^\infty}+2^q\int_0^t\big(\|v^1(\tau)\|_{L^\infty}\|\theta(\tau)\|_{L^\infty}+\|\theta^2(\tau)\|_{L^\infty}\|v(\tau)\|_{L^\infty}\big)d\tau\\ &+ 2^{2q}\|\theta\|_{L^1_tL^\infty}\end{aligned}$$

Combining both last estimates with Besov embeddings (essentially $\dot B^0_{\infty,1}\hookrightarrow B^0_{\infty,1}\hookrightarrow L^\infty$) gives

$$\begin{aligned}\|\theta(t)\|_{\dot B^0_{\infty,1}} &\lesssim \sum_{q\leq 0}\|\dot\Delta_q\theta^0\|_{L^\infty}+\int_0^t(1+w_{1,2}(\tau))\|\theta(\tau)\|_{\dot B^0_{\infty,1}}d\tau\\ &+ \|\theta^{2,0}\|_{L^\infty}\int_0^t\|v(\tau)\|_{B^0_{\infty,1}}d\tau+\|\theta(t)-\Delta_{-1}\theta(t)\|_{B^0_{\infty,1}}.\end{aligned}$$

Integrating over the time and using Gronwall's inequality

$$\begin{aligned}\|\theta\|_{L^1_t\dot B^0_{\infty,1}}\lesssim e^{Ct+Ct\|w_{1,2}\|_{L^\infty_t}}\Big(\sum_{q\leq0}\|\dot\Delta_q\theta^0\|_{L^\infty} &+ \|\theta^{2,0}\|_{L^\infty}\int_0^t\|v(\tau)\|_{B^0_{\infty,1}}d\tau\\ &+ \|\theta-\Delta_{-1}\theta\|_{L^1_tB^0_{\infty,1}}\Big).\end{aligned}\tag{31}$$

It remains to estimate $\|\theta-\Delta_{-1}\theta\|_{L^1_tB^0_{\infty,1}}$. For this purpose we apply Proposition 2.3 to the equation (29) with $s=-\frac12$ and $p=r=\infty$

$$\|\theta-\Delta_{-1}\theta\|_{L^1_tB^{\frac32}_{\infty,\infty}} \lesssim e^{V_1(t)}\Big(\|\theta^0\|_{B^{\frac{-1}{2}}_{\infty,\infty}}+\int_0^t\|v\cdot\nabla\theta^2(\tau)\|_{L^\infty}d\tau\Big).$$

We have used in the above inequality the embedding $L^\infty\hookrightarrow B^{-\frac12}_{\infty,\infty}$.
Since $B^{\frac32}_{\infty,\infty}\hookrightarrow B^0_{\infty,1}$, we find

$$\|\theta-\Delta_{-1}\theta\|_{L^1_tB^0_{\infty,\infty}} \lesssim e^{Ct\|w_{1,2}\|_{L^\infty_t}}\Big(\|\theta^0\|_{B^{\frac{-1}{2}}_{\infty,\infty}}+\int_0^t\|v(\tau)\|_{B^0_{\infty,1}}\|\nabla\theta^2(\tau)\|_{L^\infty}d\tau\Big).$$

Inserting this estimate into (31) we get

$$\begin{aligned}\|\theta\|_{L^1_t\dot B^0_{\infty,1}} &\lesssim e^{Ct+C\|w_{1,2}\|_{L^\infty_t}}\Big(\sum_{q\leq0}\|\dot\Delta_q\theta^0\|_{L^\infty}+\|\theta^0\|_{B^{\frac{-1}{2}}_{\infty,\infty}}\\ &+ \int_0^t\|v(\tau)\|_{B^0_{\infty,1}}\big(\|\nabla\theta^2(\tau)\|_{L^\infty}+\|\theta^{2,0}\|_{L^\infty}\big)d\tau\Big).\end{aligned}$$

Putting together this estimate with (30) we obtain

$$\begin{aligned}\|v(t)\|_{B^0_{\infty,1}} &\lesssim e^{Ct+Ct\|w_{1,2}\|_{L^\infty_t}}\Big(\sum_{q\leq0}\|\dot\Delta_q\theta^0\|_{L^\infty}+\|\theta^0\|_{B^{\frac{-1}{2}}_{\infty,\infty}}+\|v^0\|_{B^0_{\infty,1}}\\ &+ \int_0^t\|v(\tau)\|_{B^0_{\infty,1}}\big(w_{1,2}(\tau)+\|\nabla\theta^2(\tau)\|_{L^\infty}+\|\theta^{2,0}\|_{L^\infty}\big)d\tau\Big).\end{aligned}$$

It follows from both last estimates and Gronwall's inequality

$$\|v\|_{L^\infty_t B^0_{\infty,1}} + \|\theta\|_{L^1_t \dot B^0_{\infty,1}} \le \eta(t)\Big(\sum_{q\le 0}\|\dot\Delta_q\theta^0\|_{L^\infty} + \|\theta^0\|_{B^{\frac{-1}{2}}_{\infty,\infty}} + \|v^0\|_{B^0_{\infty,1}}\Big), \tag{32}$$

where $\eta = \mathbb{R}_+ \to \mathbb{R}_+$ is a function depending on the quantities $\|v^j\|_{L^\infty_t B^1_{\infty,1}}$, $\|\nabla\theta^j\|_{L^1_t L^\infty}$ and $\|\theta^{j,0}\|_{L^\infty}$. This concludes the proof of the uniqueness part.

5.3. **Existence part.** We will briefly outline the proof of the existence part which is classical. We smooth out the initial data $(v^0_n, \theta^0_n) := (S_n v^0, S_n\theta^0)$, which is uniformly bounded in $B^1_{\infty,1}\times\mathcal{B}^\infty$ and it is easy to check the following convergence result

$$\lim_{n\to\infty}\Big(\|v^0_n - v^0\|_{B^0_{\infty,1}} + \sum_{q\le 0}\|\dot\Delta_q(\theta^0_n-\theta^0)\|_{L^\infty} + \|\theta^0_n - \theta^0\|_{B^{\frac{-1}{2}}_{\infty,\infty}}\Big) = 0.$$

Since the initial data (v^0_n, θ^0_n) are smooth then the corresponding Boussinesq system has global unique smooth solutions (v_n, θ_n). In view of Proposition 5.1 one has for every $t\in\mathbb{R}_+$, the uniform estimates:

$$\|v_n\|_{L^\infty_t B^1_{\infty,1}} + \|\theta_n(t)\|_{L^\infty} + \|\nabla\theta_n\|_{L^1_t L^\infty} \le C_0 e^{e^{C_0 t^3}}.$$

Now according to (32) the sequence $(v_n,\theta_n)_n$ converges strongly in $L^\infty_{\text{loc}}(\mathbb{R}_+; B^0_{\infty,1}) \times L^1_{loc}(\mathbb{R}_+; \dot B^0_{\infty,1})$ to (v,θ). This is sufficiently to pass to the limit in the equations and deduce that (v,θ) satisfies the system $(B_{0,\kappa})$.
It remains to show the continuity in time of the velocity. This comes from the estimate $\|v\|_{\widetilde L^\infty_t B^1_{\infty,1}}$, (see [12] for more further details).
The proof of Theorem 1.2 is now complete.

Appendix A. Logarithmic estimate

We shall now give a logarithmic estimate which is an extension of Vishik's one [16]. Our result was firstly proved in [13] and for the convenience of the reader we will give here the proof.

Proposition 5.3. *Let p, $r\in[1,+\infty]$, v be a divergence-free vector field belonging to the space $L^1_{loc}(\mathbb{R}_+;\mathrm{Lip}(\mathbb{R}^d))$ and let a be a smooth solution of the following equation (with $\nu\ge 0$),*

$$\begin{cases} \partial_t a + v\cdot\nabla a - \nu\Delta a = f\\ a_{|t=0} = a^0.\end{cases}$$

If the initial data $a^0\in B^0_{p,r}$, then we have for all $t\in\mathbb{R}_+$

$$\|a\|_{\widetilde L^\infty_t B^0_{p,r}} \le C\big(\|a^0\|_{B^0_{p,r}} + \|f\|_{\widetilde L^1_t B^0_{p,r}}\big)\Big(1+\int_0^t \|\nabla v(\tau)\|_{L^\infty}d\tau\Big),$$

where C depends only on the dimension d but not on the viscosity ν.

Proof. We denote by $\tilde a_q$ the unique global solution of the initial value problem:

$$\begin{cases} \partial_t \tilde a_q + v\cdot\nabla\tilde a_q - \Delta\tilde a_q = \Delta_q f := f_q\\ \tilde a_q(0) = \Delta_q a^0.\end{cases}$$

Using Proposition 2.3 with $r = +\infty$ and $s = \pm\frac{1}{2}$, one obtains

$$\|\tilde{a}_q(t)\|_{B^{\pm\frac{1}{2}}_{p,\infty}} \lesssim \Big(\|\Delta_q a^0\|_{B^{\pm\frac{1}{2}}_{p,\infty}} + \int_0^t \|f_q(\tau)\|_{B^{\pm\frac{1}{2}}_{p,\infty}} d\tau\Big) e^{C\int_0^t \|\nabla v(\tau)\|_{L^\infty} d\tau}.$$

Thus we deduce from the definition of Besov spaces that for all $j \geq -1$

$$\|\Delta_j \tilde{a}_q\|_{L^\infty_t L^p} \lesssim 2^{-\frac{1}{2}|q-j|}\Big(\|\Delta_q a^0\|_{L^p} + \|f_q\|_{L^1_t L^p}\Big) e^{V(t)}, \tag{33}$$

with $V(t) := C\int_0^t \|\nabla v(\tau)\|_{L^\infty} d\tau$. Now by linearity one can write

$$a(t,x) = \sum_{q\geq -1} \tilde{a}_q(t,x).$$

Taking $N \in \mathbb{N}$ that will be carefully chosen later. Then we write by definition

$$\begin{aligned}
\|a\|_{\widetilde{L}^\infty_t B^0_{p,r}} &\leq \Big(\sum_j \Big(\sum_q \|\Delta_j \tilde{a}_q\|_{L^\infty_t L^p}\Big)^r\Big)^{\frac{1}{r}} \\
&\leq \Big(\sum_j \Big(\sum_{|q-j|\geq N} \|\Delta_j \tilde{a}_q\|_{L^\infty_t L^p}\Big)^r\Big)^{\frac{1}{r}} + \Big(\sum_j \Big(\sum_{|q-j|<N} \|\Delta_j \tilde{a}_q\|_{L^\infty_t L^p}\Big)^r\Big)^{\frac{1}{r}} \\
&= \mathrm{I} + \mathrm{II}. \qquad (34)
\end{aligned}$$

To estimate the first term we use (33) and the convolution inequality

$$\begin{aligned}
\mathrm{I} &\lesssim 2^{-\frac{1}{2}N} e^{V(t)} \|\big(\|{a_q}^0\|_{L^p} + \|f_q\|_{L^1_t L^p}\big)_q\|_{\ell^r} \\
&\lesssim 2^{-\frac{1}{2}N} e^{V(t)} \Big(\|a^0\|_{B^0_{p,r}} + \|f\|_{\widetilde{L}^1_t B^0_{p,r}}\Big). \qquad (35)
\end{aligned}$$

To treat the second term of the right-hand side of (34), we use two facts: the first one is that the operator Δ_j maps uniformly L^p into itself while the second is the L^p energy estimate. So we find

$$\begin{aligned}
\mathrm{II} &\lesssim \Big(\sum_j \Big(\sum_{|q-j|<N} \|\tilde{a}_q\|_{L^\infty_t L^p}\Big)^r\Big)^{\frac{1}{r}} \\
&\lesssim \Big(\sum_j \Big(\sum_{|q-j|<N} \|a^0_q\|_{L^p} + \|f_q\|_{L^1_t L^p}\Big)^r\Big)^{\frac{1}{r}} \\
&\lesssim N\big(\|a^0\|_{B^0_{p,r}} + \|f\|_{\widetilde{L}^1_t B^0_{p,r}}\big). \qquad (36)
\end{aligned}$$

Plugging estimates (35) and (36) into (34), we have

$$\|a\|_{\widetilde{L}^\infty_t B^0_{p,r}} \lesssim \big(\|a^0\|_{B^0_{p,r}} + \|f\|_{\widetilde{L}^1_t B^0_{p,r}}\big)\big(2^{-\frac{1}{2}N} e^{V(t)} + N\big).$$

Taking

$$N = \Big[\frac{2V(t)}{\log 2} + 1\Big],$$

leads to the desired inequality. $\square$

Appendix B. Commutator estimate

Our task now is to prove the following commutator result.

Proposition 5.4. *Let u be a smooth function and v be a divergence-free vector field of $\mathbb{R}^d$ such that its vorticity $\omega := curl\ v$ belongs to L^∞. Then we have for all $q \geq -1$,*

$$\big\|[\Delta_q, v\cdot\nabla]u\big\|_{L^\infty} \lesssim \|u\|_{L^\infty}\Big(\|\nabla\Delta_{-1}v\|_{L^\infty} + (q+2)\|\omega\|_{L^\infty}\Big).$$

Proof. The principal tool is Bony's decomposition [2]:

$$[\Delta_q, v\cdot\nabla]u = [\Delta_q, T_v\cdot\nabla]u + [\Delta_q, T_{\nabla\cdot}\cdot v]u + [\Delta_q, R(v\cdot\nabla, .)]u, \tag{37}$$

where

$$\begin{aligned}
[\Delta_q, T_v\cdot\nabla]u &= \Delta_q(T_v\cdot\nabla u) - T_v\cdot\nabla\Delta_q u\\
[\Delta_q, T_{\nabla\cdot}\cdot v]u &= \Delta_q(T_{\nabla u}\cdot v) - T_{\nabla\Delta_q u}\cdot v\\
[\Delta_q, R(v\cdot\nabla, .)]u &= \Delta_q(R(v\cdot\nabla, u)) - R(v\cdot\nabla, \Delta_q u).
\end{aligned}$$

From the definition of the paraproduct and according to Bernstein inequalities

$$\begin{aligned}
\|[\Delta_q, T_{\nabla\cdot}\cdot v]u\|_{L^\infty} &\lesssim \sum_{|j-q|\leq 4}\|S_{j-1}\nabla u\|_{L^\infty}\|\Delta_j v\|_{L^\infty}\\
&\lesssim \|u\|_{L^\infty}\|\omega\|_{L^\infty}, \qquad (38)
\end{aligned}$$

where we have used here the following equivalence: $\forall j\in\mathbb{N}$,

$$\|\Delta_j v\|_{L^\infty} \approx 2^{-j}\|\Delta_j\omega\|_{L^\infty}.$$

For the second term of the right-hand side of (37), we have

$$\begin{aligned}
[\Delta_q, T_v\cdot\nabla]u &= \sum_{j\geq 1}[\Delta_q, S_{j-1}v\cdot\nabla\Delta_j]u,\\
&= \sum_{|j-q|\leq 4}[\Delta_q, S_{j-1}v\cdot\nabla]\Delta_j u.
\end{aligned}$$

To estimate each commutator, we write Δ_q as a convolution

$$[\Delta_q, S_{j-1}v\cdot\nabla]\Delta_j u(x) = 2^{qd}\int h(2^q(x-y))\big(S_{j-1}v(y) - S_{j-1}v(x)\big)\cdot\nabla\Delta_j u(y)dy.$$

Thus, Young and Bernstein inequalities yield, for $|j-q|\leq 4$,

$$\begin{aligned}
\big\|[\Delta_q, S_{j-1}v\cdot\nabla]\Delta_j u\big\|_{L^\infty} &\lesssim 2^{-q}\|\nabla S_{j-1}v\|_{L^\infty}\|\Delta_j\nabla u\|_{L^\infty}\\
&\lesssim \|\nabla S_{j-1}v\|_{L^\infty}\|u\|_{L^\infty}\\
&\lesssim \Big(\|\nabla\Delta_{-1}v\|_{L^\infty} + (q+2)\|\omega\|_{L^\infty}\Big)\|u\|_{L^\infty}.
\end{aligned} \tag{39}$$

Let us move to the remainder term. It can be written, in view of the definition, as

$$\mathrm{J}_q := [\Delta_q, R(v\cdot\nabla, .)]u = \sum_{\substack{j\geq q-4, j\geq 0\\ i\in\{\mp 1, 0\}}}[\Delta_q, \Delta_j v]\cdot\nabla\Delta_{j+i}u + \sum_{i\in\{0,1\}}[\Delta_q, \Delta_{-1}v]\cdot\nabla\Delta_{-1+i}u.$$

It follows from the zero divergence condition that

$$\mathrm{J}_q=\sum_{i\in\{0,1\}}[\Delta_q,\Delta_{-1}v]\cdot\nabla\Delta_{-1+i}u+\sum_{\substack{j\geq q-4, j\geq 0\\ i\in\{\mp1,0\}}}\mathrm{div}\,\big([\Delta_q,\Delta_j v]\otimes\Delta_{j+i}u\big)=\mathrm{J}_q^1+\mathrm{J}_q^2.$$

By the same way as (39) one has

$$\begin{aligned}\|\mathrm{J}_q^1\|_{L^2}&\lesssim 2^{-q}\|\nabla\Delta_{-1}v\|_{L^\infty}\sum_{i=0}^{1}\|\nabla\Delta_{-1+i}u\|_{L^\infty}\\&\lesssim\|\nabla\Delta_{-1}v\|_{L^\infty}\|u\|_{L^\infty}.\end{aligned}$$

To estimate the second term we use Bernstein inequality

$$\begin{aligned}\|\mathrm{J}_q^2\|_{L^2}&\lesssim\sum_{\substack{j\geq q-4, j\geq 0\\ i\in\{\mp1,0\}}}2^q\|\Delta_j v\|_{L^\infty}\|\Delta_{j+i}u\|_{L^\infty}\\&\lesssim\|u\|_{L^\infty}\sum_{j\geq q-4}2^{q-j}\|\Delta_j\omega\|_{L^\infty}\\&\lesssim\|\omega\|_{L^\infty}\|u\|_{L^\infty},\end{aligned}$$

This completes the proof of Proposition 5.4. $\square$

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